

ADIABATIC LIMITS ON RIEMANNIAN SOL-MANIFOLDS

ANDREY A. YAKOVLEV

ABSTRACT. We obtain an asymptotic formula for the spectrum distribution function of the Laplace operator on a compact Riemannian Sol-manifold in the adiabatic limit determined by a one-dimensional foliation defined by the orbits of a left-invariant flow.

The paper is devoted to investigation of adiabatic limits on Riemannian Sol-manifolds. We understand adiabatic limits in the sense, which was introduced by Witten in [1]. More precisely, let (M, \mathcal{F}) be a closed foliated manifold equipped with a Riemannian metric g . Thus, the tangent bundle TM of M is represented as a direct sum

$$TM = F \oplus H,$$

where $F = T\mathcal{F}$ is the tangent bundle of \mathcal{F} and $H = F^\perp$ the orthogonal complement of F . Let g_F and g_H denote the restriction of the metric g to F and H , respectively. Therefore, $g = g_F + g_H$. Define a one-parameter family of Riemannian metrics on M by the formula

$$(1) \quad g_\varepsilon = g_F + \varepsilon^{-2} g_H, \quad \varepsilon > 0.$$

Investigation of various properties of the family of Riemannian manifolds (M, g_ε) as $\varepsilon \rightarrow 0$ will be called by passage to adiabatic limit.

Recall [2] that the group Sol is the solvable Lie subgroup of the Lie group $GL(3, \mathbb{R})$, which consists of all matrices of the form:

$$\gamma(u, v, w) = \begin{pmatrix} e^w & 0 & u \\ 0 & e^{-w} & v \\ 0 & 0 & 1 \end{pmatrix}, \quad (u, v, w) \in \mathbb{R}^3.$$

The Lie algebra \mathfrak{sol} of Sol is the Lie subalgebra of the Lie algebra $gl(3, \mathbb{R})$, which consists of all matrices of the form

$$X(u, v, w) = \begin{pmatrix} w & 0 & u \\ 0 & -w & v \\ 0 & 0 & 0 \end{pmatrix}, \quad (u, v, w) \in \mathbb{R}^3.$$

Let $A \in SL(2, \mathbb{Z})$ and $|\operatorname{tr} A| > 2$. Denote by λ and λ^{-1} the eigenvalues of A and assume that $\lambda > 1$. Define a vectors $(c_1^1, c_1^2), (c_2^1, c_2^2)$ by the equation

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix}^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix}.$$

Supported by the Russian Foundation of Basic Research (grant no. 06-01-00208).

Definition 1. A Riemannian Sol-manifold is a compact manifold $M_A^3 = G_A \backslash \text{Sol}$ equipped with a Riemannian metric g , where:

- G_A is the uniform discrete subgroup of the Lie group Sol , which consists of all $\gamma(u, v, w) \in \text{Sol}$ such that

$$(u, v) \in \Gamma := \{k(c_1^1, c_1^2) + l(c_2^1, c_2^2), \quad k, l \in \mathbb{Z}\},$$

$$w = m \ln \lambda, \quad m \in \mathbb{Z},$$

- g is a Riemannian metric on M_A^3 whose lift on Sol is invariant under left translations by elements of Sol (such metrics will be called locally left-invariant).

A locally left-invariant metric g is uniquely determined by its value at the identity $\gamma(0, 0, 0)$ of Sol , and, therefore, is given by a symmetric positive definite 3×3 -matrix.

Let $\alpha \in \mathbb{R}$. Consider the left-invariant vector field on Sol associated with $X(1, \alpha, 0) \in \mathfrak{sol}$. The orbits of the corresponding vector field on M_A^3 define a one-dimensional foliation \mathcal{F} . The leaf of \mathcal{F} through $G_A \gamma(u, v, w) \in M_A^3$ is given by

$$L_{G_A \gamma(u, v, w)} = \{G_A \gamma(u + e^w t, v + \alpha e^{-w} t, w) \in M_A^3 : t \in \mathbb{R}\}.$$

Suppose that a locally left-invariant metric g correspond to the identity matrix. Consider the adiabatic limit associated with the Riemannian Sol-manifold (M_A^3, g) and the foliation \mathcal{F} . Denote by Δ_ε the Laplace-Beltrami operator on M_A^3 associated with the metric g_ε given by (1). For any $\varepsilon > 0$ the spectrum of Δ_ε consists of eigenvalues of finite multiplicity:

$$0 = \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \leq \dots, \lambda_j(\varepsilon) \rightarrow +\infty \quad j \rightarrow \infty.$$

The main result of the paper is a computation of the asymptotics of the spectrum distribution function

$$N_\varepsilon(t) = \sharp\{i : \lambda_i(\varepsilon) \leq t\}$$

of the operator Δ_ε in the adiabatic limit, that is, when $t \in \mathbb{R}$ is fixed and $\varepsilon \rightarrow 0$.

Theorem 2. *For any $t > 0$, the following asymptotic formulae hold:*

1. *For $\alpha \neq 0$*

$$N_\varepsilon(t) = \frac{1}{4\pi^2} t^{\frac{3}{2}} \varepsilon^{-2} + o(\varepsilon^{-2}), \quad \varepsilon \rightarrow 0.$$

2. *For $\alpha = 0$*

$$N_\varepsilon(t) = \frac{1}{6\pi^2} t^{\frac{3}{2}} \varepsilon^{-2} + o(\varepsilon^{-2}), \quad \varepsilon \rightarrow 0.$$

The asymptotic behavior of the spectrum distribution function for the Laplace operator in the adiabatic limit was studied earlier in [3] for Riemannian foliations and in [4] for one-dimensional foliations on Riemannian Heisenberg manifolds (see also [5]). In all cases, the function $N_\varepsilon(t)$ has order ε^{-q} , where q is the codimension of the foliation (in our case $q = 2$), but the coefficients of ε^{-q} are different in each case. Observe also that, in each case, the asymptotic formula for $N_\varepsilon(t)$ in the adiabatic limit is different from the classical Weyl formula, which describes asymptotic behavior of $N_\varepsilon(t)$ as $t \rightarrow \infty$ (cf. [6]).

For $\alpha \neq 0$, the proof of the theorem uses the calculation of the spectrum of the Laplace operator on a Riemannian Sol-manifold given in [6], which continues the investigation of the geodesic flow on a Riemannian Sol-manifold started in [7] and [8], and semiclassical spectral asymptotics [9] for the modified Mathieu operator

$$H_\varepsilon = -\varepsilon^2 \frac{d^2}{dx^2} + a \operatorname{ch}(2\mu x), \quad x \in \mathbb{R}.$$

In the case $\alpha = 0$, the foliation is Riemannian, and the metric is bundle-like, and, therefore, we can use the asymptotic formula obtained in [3].

The author is grateful to Yu.A. Kordyukov for posing this problem and attention to his work and to I.A. Taimanov for useful remarks.

REFERENCES

- [1] E. Witten, Global gravitational anomalies, *Comm. Math. Phys.* **100** (1985), 197–229.
- [2] W. Thurston, Hyperbolic geometry and 3-manifolds, *Low-dimensional topology (Bangor, 1979)*, Cambridge Univ. Press, Cambridge-New York, 1982, 9–25.
- [3] Yu. A. Kordyukov, Adiabatic limits and spectral geometry of foliations, *Math. Ann.* **313** (1999), 763–783.
- [4] A.A. Yakovlev, Adiabatic limits on Riemannian Heisenberg manifolds, *Mat. sb.* **199**, no.2 (2008), 149–160.
- [5] Yu.A. Kordyukov, A.A. Yakovlev, Adiabatic limits and the spectrum of the Laplacian on foliated manifolds, *C*-algebras and elliptic theory. II*, Trends in Mathematics, 123 – 144, Birkhäuser, Basel, 2008; preprint math.DG/0703785.
- [6] A.V. Bolsinov, H.R. Dullin, A.P. Veselov, Spectra of Sol-manifolds: arithmetic and quantum monodromy, *Comm. Math. Phys.* **264** (2006), 583–611.
- [7] A.V. Bolsinov, I.A. Taimanov, Integrable geodesic flows with positive topological entropy, *Invent. Math.* **140** (2000), 639–650.
- [8] A.V. Bolsinov, I.A. Taimanov, Integrable geodesic flows on suspensions of automorphisms of tori. (Russian) *Tr. Mat. Inst. Steklova* **231** (2000), Din. Sist., Avtom. i Beskon. Gruppy, 46–63; translation in *Proc. Steklov Inst. Math.* 2000, no. 4 (231), 42–58.
- [9] B. Helffer, A. Martinez, D. Robert, Ergodicite et limite semi-classique, *Comm. Math. Phys.* **109** (1987), 313–326.

DEPARTMENT OF MATHEMATICS, UFA STATE AVIATION TECHNICAL UNIVERSITY, 12
K. MARX STR., 450000 UFA, RUSSIA
E-mail address: yakovlevandrey@yandex.ru